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Minimum Equitable Dominating Harary Energy of a Graph

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Abstract

The purpose of this paper is to introduce the concept of minimum equitable dominating Harary energy $HE_{ED}(G)$ of a graph. We also compute the minimum equitable dominating Harary energy $HE_{ED}(G)$ of some families of graphs. We establish bounds for minimum equitable dominating Harary energy. Open problems are also given for further research.

Keywords: Minimum equitable dominating set, minimum equitable dominating Harary eigenvalues, minimum equitable dominating Harary energy of a graph.
2010 Mathematics subject classification: 05C15, 05C50

1 Introduction

Through out the paper we consider simple, undirected, signless, markless graphs. For a simple graph G of order n with vertex set $V = \{v_1, v_2, v_3, \cdots, v_n\}$ and edge set E, the distance between the vertices v_i and v_j, denoted by $d_{ij} = d(v_i, v_j)$ is the length of the shortest path joining them. The Harary matrix, [4], of a graph G is an $n \times n$ matrix (h_{ij}) given by

$$h_{ij} = \begin{cases} \frac{1}{d_{ij}}, & \text{if } i \neq j \\ 0, & \text{otherwise.} \end{cases}$$

A subset U of $V(G)$ is an equitable dominating set if for every $v \in V(G) - U$, there exists a vertex $u \in U$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$ where $deg(x)$ denotes the degree of vertex x in $V(G)$. Any equitable dominating set with minimum cardinality is called a minimum equitable dominating set. For a

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minimum equitable dominating set ED of a graph G, the minimum equitable dominating Harary matrix is given by $H_{ED}(G) = (hed_{ij})$ where

$$hed_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } v_i \in D \\ 0, & \text{if } i = j \text{ and } v_i \notin D \\ \frac{1}{d_{ij}}, & otherwise \end{cases}.$$

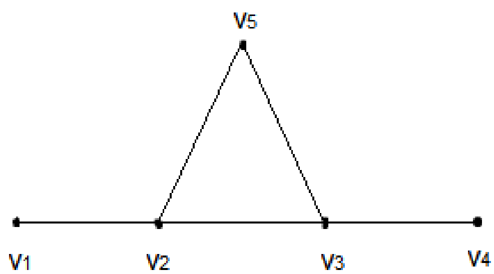
The minimum equitable dominating Harary energy of G is defined in a similar fashion to classical graph energy, see [3, 10], as

$$HE_{ED}(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

Some mathematical properties of graph energy and some special energies are studied in [5, 9, 15, 17, 18, 21] and some chemical properties of them are studied in [7, 11, 12, 13]. Several lower and upper bounds on these energy types are studied in [1, 2, 14, 16]. All studies in graph energies depend on some type of graph matrices which were clearly catalogued in [4]. The area of graph theory dealing with graph energies is called as spectral graph theory, see [6, 8, 19, 20]. In the remaining part of this paper, we study the minimum equitable dominating Harary energy and obtain few properties of it.

2 An example

Let G be the graph as shown in the diagram below. For G, a minimum equitable dominating set is $D = \{v_2, v_3\}$. Then



Here the minimum equitable dominating Harary matrix is given by

$$H_{ED}(G) = \begin{bmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 1 & 1 & 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & 1 & 1 & 1 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 & \frac{1}{2} & 0 \end{bmatrix}.$$

The characteristic polynomial of H_{ED} is $\lambda^5-2\lambda^4-5.111\lambda^3-2.111\lambda^2+0.5764\lambda+0.3333$ and therefore the spectrum is

$$spec_{ED}(G)=\begin{pmatrix}-0.7452 & -0.6937 & -0.5 & 0.3604 & 3.5785\\ 1 & 1 & 1 & 1 & 1\end{pmatrix}.$$

Hence the minimum equitable dominating Harary energy would be $HE_{ED}(G)=5.8778$.

3 Minimum equitable dominating Harary energy of some standard graphs

Theorem 3.1. *The minimum equitable dominating Harary energy of the complete bipartite graph $K_{n\times n}$ is*

$$HE_{ED}(K_{n\times n})=\frac{3n}{2}+(n-2)+\frac{\sqrt{n^2+4n-4}}{2}$$

Proof. Let $K_{n\times n}$ be the complete bipartite graph of order $n\times n$ with vertex set $\{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$. The minimum equitable dominating Harary matrix is

$$\begin{bmatrix}
1 & 1/2 & 1/2 & \dots & 1/2 & 1 & 1 & \dots & 1 & 1\\
1/2 & 0 & 1/2 & \dots & 1/2 & 1 & 1 & \dots & 1 & 1\\
1/2 & 1/2 & 0 & \dots & 1/2 & 1 & 1 & \dots & 1 & 1\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1/2 & 1/2 & 1/2 & \dots & 0 & 1 & 1 & \dots & 1 & 1\\
1 & 1 & 1 & \dots & 1 & 1 & 1/2 & 1/2 & \dots & 1/2\\
1 & 1 & 1 & \dots & 1 & 1/2 & 0 & 1/2 & \dots & 1/2\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 1 & 1 & \dots & 1 & 1/2 & 1/2 & 1/2 & \dots & 1/2\\
1 & 1 & 1 & \dots & 1 & 1/2 & 1/2 & 1/2 & \dots & 1/2
\end{bmatrix}.$$

In that case, the characteristic equation is

$$(\lambda+1/2)^{2n-4}\left(\lambda^2-\frac{3n}{2}\lambda+\frac{3n-7}{4}\right)^{n-1}\left(\lambda^2+\frac{n}{2}\lambda-\frac{n-1}{4}\right)=0$$

implying that the spectrum is

$$\begin{pmatrix}\frac{-1}{2} & \frac{3n+\sqrt{9n^2-12n+28}}{4} & \frac{3n-\sqrt{9n^2-12n+28}}{4} & \frac{-n+\sqrt{n^2+4n-4}}{4} & \frac{-n-\sqrt{n^2+4n-4}}{4}\\ 2n-4 & 1 & 1 & 1 & 1\end{pmatrix}.$$

Therefore,

$$HE(S_n^0)=\frac{3n}{2}+(n-2)+\frac{\sqrt{n^2+4n-4}}{2}.$$

□

Theorem 3.2. *The minimum equitable dominating Harary energy of the crown graph S_n^0 is*

$$RSE(S_n^0) = \frac{1}{6}\left[8n - 16 + \sqrt{n^2 + 4n - 4} + \sqrt{9n^2 - 12n + 18}\right].$$

Proof. The minimum equitable dominating set is $\{v_1, v_3 ... v_n\}$. The same proof technique will give the result. □

Theorem 3.3. $HE_{ED}(K_n) = (n - 2) + \sqrt{n^2 - 2n + 5}$.

Proof. For complete graph K_n, the vertex set is $V = \{v_1, v_2, \cdots, v_n\}$ and minimum equitable dominating set is $D = \{v_1\}$. The same proof technique will give the result. □

Theorem 3.4. *The minimum equitable dominating Harary energy of the complement $\overline{K_n}$ of the complete graph is*

$$RSE\overline{(K_n)} = 0.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1,\ v_2,\ \cdots,\ v_n\}$. Since the minimum equitable dominating set has no vertex, the matrix is

$$HE_{ED}(\overline{K_n}) = \left[0\right]_{n\times n}.$$

Then the characteristic equation is $\lambda^n = 0$. Therefore,

$$HE_{ED}(\overline{K_n}) = 0.$$

□

Theorem 3.5. *If $K_{1,n-1}$ is a star graph of order $n \geq 3$, then*

$$HE_{ED}(K_{1,n-1}) = \frac{1}{2}\left[n - 2 + \sqrt{n^2 + 8n}\right].$$

Proof. $K_{1,n-1}$ has a minimum equitable dominating set $D = \{v_0\}$. The same proof technique can be used to prove this. □

Theorem 3.6. *If $K_{1,n-1}$ is a star graph of order $n \geq 3$ and $\overline{K_{1,n-1}}$ be it's complement, then*

$$HE_{ED}(\overline{K_{1,n-1}}) = (n - 3) + \sqrt{n^2 - 4n + 8}.$$

Proof. $\overline{K_{1,n-1}}$ has a minimum equitable dominating set $D = \{v_0\}$. Then

$$H_{ED}(\overline{K_{1,n-1}}) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \dots & \frac{1}{2} & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{1}{2} & \frac{1}{2} & \dots & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & 0 \end{bmatrix}_{n\times n}.$$

Characteristic equation for $n \geq 3$ is $\lambda(\lambda+1)^{n-3}(\lambda^2-(n-2)\lambda-1)=0$ and the minimum equitable dominating Harary eigenvalues for $n \geq 3$ are

$$spec_{ED}\overline{K_{1,n-1}} = \begin{pmatrix} -1 & 0 & \frac{n-2+\sqrt{n^2-4n+2}}{2} & \frac{n-2+\sqrt{n^2-4n+2}}{2} \\ n-3 & 1 \quad 1 & 1 & \end{pmatrix}.$$

As a result, the minimum equitable dominating Harary energy is

$$(n-3)+\sqrt{n^2-4n+8}$$

□

Recall that a cocktail party graph is denoted by $K_{n\times 2}$ and is a graph having vertex set $V = \cup_{i=1}^{n}\{u_i, v_i\}$ and edge set $E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \leq i < j \leq n\}$. This graph is also called as a complete n-partite graph. The following graphs also have the minimum dominating set equal to their minimum equitable dominating set, thus we ommit the proof.

Theorem 3.7. *If $K_{n\times 2}$ is a cocktail party graph of order $2n$, then $HE_{ED}(K_{n\times 2}) = 2n-3+\sqrt{4n^2-4n+9}$.*

The friendship graph is denoted by $F_3^{(n)}$ and is a graph obtained by taking n copies of the cycle graph C_3 with a vertex in common. Then we have the following result.

Theorem 3.8. $HE_{ED}(F_3^{(n)}) = n+\sqrt{n^2+6n+1}$.

Theorem 3.9. *The minimum equitable dominating Harary energy of the complement $\overline{K_{n\times 2}}$ of the cocktail party graph $K_{n\times 2}$ of order $2n$ is*

$$HE_{ED}\overline{(K_{n\times 2})} = 2n-2+\sqrt{5}.$$

Proof. Let $\overline{(K_{n\times 2})}$ be the cocktail party graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The matrix is

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Characteristic equation is

$$(\lambda^2-1)^{n-1}(\lambda^2-\lambda-1)=0$$

Hence, spectrum is

$$\begin{pmatrix} 1 & -1 & 1.618 & -0.618 \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

Therefore, $HE_{ED}(\overline{K_{n\times 2}}) = 2n-2+\sqrt{5}$. □

4 Properties of minimum equitable dominating Harary energy of a graph

We have different domination parameters like minimum dominating, global dominating, secure dominating etc., Let X be any such domination set X, in this section we give some properties with respect to general dominating set X. Let $|\lambda I - H_X| = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$ be the characteristic polynomial of H_X.

Theorem 4.1. *Let X be any dominating set (for ex. minimum dominating or boundary dominating etc.,) then, the first three coefficients of characteristic polynomial are*
(i) $a_0 = 1$,
(ii) $a_1 = -Cardinality\ of\ X$,
(iii) $a_2 = \binom{X}{2} - \sum_{i<j}^{n} \frac{1}{d(v_i, v_j)^2}$.

Proof. (i) Using the characteristic equation, we get $a_0 = 1$.
(ii) The sum of determinants of all 1×1 principal submatrices of H_{ED} is equal to the trace of H_X. This implies that $a_1 = (-1)^1 \cdot$ trace of $[H_X(G)] = -Cardinality\ of\ X$.
(iii) The sum of determinants of all the 2×2 principal submatrices of $[H_X(G)]$ is

$$\begin{aligned} a_2 = (-1)^2 \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ij}a_{ji} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ji}a_{ij} \\ &= \binom{X}{2} - \sum_{i<j}^{n} \frac{1}{d(v_i, v_j)^2}. \end{aligned}$$

□

Theorem 4.2. *Let G be a graph with n vertices m edges and minimum equitable dominating set ED. Then* $\sqrt{|X| + 2\sum_{i<j}^{n} \frac{1}{d(v_i, v_j)^2} + n(n-1)|detA_X(G)|^{\frac{2}{n}}} \le HE_X(G) \le \sqrt{n\left(|X| + 2\sum_{i<j}^{n} \frac{1}{d(v_i, v_j)^2}\right)}$.

Proof. Upper bound: It can be easily proven by the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2,$$

The substitutions $a_i = 1$ and $b_i = |\lambda_i|$ yield the upper bound with respect to general dominating set X.

$$[HE_X] \leq \sqrt{n\left(|X| + 2\sum_{i<j}^{n} \frac{1}{d(v_i, v_j)^2}\right)}$$

Lower bound: It can be easily proven by using arithmetic mean and geometric mean inequality. □

5 Open Problems

Open problem 5.1. *Obtain the class of graphs which are equienergitic with respect to minimum equitable dominating Harary matrix.*

Open problem 5.2. *Study the properties of Laplacian minimum equitable dominating Harary matrix.*

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